The direct link model for polymer rings using topological field theory and the second topological moment in dense systems

2001 J. Phys. A: Math. Gen. 34 2539

(http://iopscience.iop.org/0305-4470/34/12/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 2539–2547 www.iop.org/Journals/ja PII: S0305-4470(01)17971-0

The direct link model for polymer rings using topological field theory and the second topological moment in dense systems

Matthias Otto[1]

Institut für Theoretische Physik, Universität Göttingen, Bunsenstrasse 9, D-37073 Göttingen, Germany

Received 9 October 2000, in final form 16 January 2001

Abstract

Polymer rings in solution are either permanently entangled or are not. Permanent topological restrictions give rise to additional entropic interactions apart from the ones arising due to mere chain flexibility or excluded volume. Conversely, entangled polymer rings systems may be formed by closing randomly entangled flexible linear chains. The dependence of linking numbers between randomly entangled rings on the chain length, more specifically the second topological moment $\langle n^2 \rangle$, i.e. the average squared linking number, may be determined. In this paper, an approach recently discussed in mathematical physics and called Abelian BF theory, is presented which allows one to express the linking constraint in its simplest form, the Gauss integral, in terms of two gauge fields. The model of Brereton and Shah for a single ring entangled with many other surrounding rings is rederived. The latter model is finally used to calculate the second topological moment, in agreement with a recent result by Ferrari, Kleinert, and Lazzizzera obtained using n-component ϕ^4 theory for the limit $n \to 0$.

PACS numbers: 0240, 0520, 0540F, 1110, 1115

1. Introduction

The topology of knots and links has attracted the interest of mathematicians and physicists alike (see for a general introduction e.g. [1–5, 7]). In polymer physics, the role of topological constraints has been discussed since its beginnings [8, 9]. On the level of a self-avoiding walk description for the polymer conformation, the Gauss invariant (defined below) was discussed by Edwards who also considered the role of higher order link invariants (HOLC). In [10], the Abelian Chern–Simons theory (however, not as per the present terminology which is taken from the authors of [11]) was used to express the self-linking number of a single ring. In his discussion of HOLC, Edwards pointed to the role of a three-vertex

[1] Present address: SPEC-CEA, Orme des Merisiers, F-91191 Gif sur Yvette, France.

0305-4470/01/122539+09$30.00

connecting three gauge fields, a remark anticipating the non-Abelian Chern–Simons theory which was used by Witten [12] some 20 years later to obtain polynomial invariants for knots and links. These algebraic invariants which are usually defined by mathematical knot theorists in terms of recursion relations (skein relations) with respect to the crossings appearing in the two-dimensional projections of knots and links have also been applied to polymers (for this important approach which will not be explored here the reader is referred to [7]). The conjecture by Edwards was validated, for example, by the work of Cotta-Ramusino, Guadagnini *et al* [6] (see [2,7] and references therein) and by Axelrod *et al* [14] and Alvarez and Labastida (see [15] and references therein), who showed that the Gauss invariant and all HOLC appear in a perturbation series of the non-Abelian Chern–Simons theory when the vacuum expectation value of products of Wilson loops is evaluated. This result suggests one should treat the Gauss invariant as an approximate measure of true topology.

The concept of the Abelian variant of Chern–Simons field theory has been reconsidered by many authors detailed in Refs [13,16–20], as well as for many rings systems, since it arises naturally whenever the Gaussian linking number is enforced as a constraint in the partition function. Partial reviews of the subject are found in Refs [4, 5, 7]. The non-Abelian variant has been discussed only recently by some authors [4, 13, 17], however, its implementation in terms of a well motivated physical model for entangled polymers is not so clear.

Whenever the Abelian Chern–Simons theory is used in the context of the Gauss integral, an important problem arises in terms of self-linking integrals, where the double line integration is not taken with respect to two different ring contours but with respect to a single ring contour. These integrals attain a meaning only when using the so-called vertical framing, i.e. by replacing the single ring by a band. For details of this and the relevant literature we refer the reader to [2].

Following a suggestion of Thompson (see e.g. the review [22]), an alternative topological quantum field theory, termed BF theory[2], is used in the present paper. In contrast to the Chern–Simons theory, it avoids the appearance of self-linking integrals, and was introduced, at least to the knowledge of the author, to the subject of polymer topology in [17]. Recently, BF theory was rediscovered in [20,21] which also dealt with avoiding self-linking integrals. In agreement with the existing literature on topological quantum field theory as in [22], the term BF theory will be used to underline the difference to the Chern–Simons theory as explained above.

In the present paper, the direct link model for a system of n_p rings of [23] is used. It singles out a single ring, say C_1 and enforces a linking number n of this test ring with all other rings $C_\beta \in \{C_2, \ldots, C_{n_p}\}$ in the system. The Gauss invariant expressing the linking number as a double line integral is then represented in terms of an Abelian BF theory. Next, the conformational degrees of freedom of all rings but C_1 are summed, giving rise to a 'mass term' to be added to the BF theory. Then, one gauge field is integrated, and the field theory of Brereton and Shah [23] is obtained, expressed in terms of the one remaining gauge field and the conformational field $\boldsymbol{r}^1(s)$, where s parametrizes the polymer contour. This theory is then used to evaluate what has recently been called the second topological moment [20], $\langle n^2 \rangle$. Essentially, the purpose of this paper is two-fold: one, to inform polymer physicists of a new approach (BF theory) for tackling entangled polymers, and two, to present a very simple derivation of the second topological moment in dense systems.

The outline of the paper is as follows. In section 2, the direct link model is presented and expressed as an Abelian BF theory. The model of Brereton and Shah is derived. In section 3,

[2] The term 'BF' originates from its non-Abelian form whose general action is given by the (wedge) product of a field $\boldsymbol{B}$ and field tensor $\boldsymbol{F_A}$ (corresponding to another field $\boldsymbol{A}$), $S = \int \mathrm{Tr}(\boldsymbol{B} \wedge \boldsymbol{F_A})$. In the Abelian case, $\boldsymbol{F_A} = \mathrm{d}\boldsymbol{A}$.

the second topological moment is evaluated. Finally, in the concluding section any relation to similar work is discussed and a simple scaling argument is evoked to interpret the results.

2. The direct link model as a BF theory

Let the conformation of the single test ring and all other rings be given by their coordinates $\{\boldsymbol{r}^1(s)\}$ and $\{\boldsymbol{r}^\beta(s)\}$ where $\beta = 2, \ldots, n_p$, respectively. Now let n be the linking number of the ring C_1 with all other rings C_β

$$n = \sum_{\beta\neq 1}^{n_p} \Phi(C_1, C_\beta) \tag{1}$$

where

$$\Phi(C_\alpha, C_\beta) = \frac{1}{4\pi} \oint_{C_\alpha} \oint_{C_\beta} \mathrm{d}\boldsymbol{r}^\alpha \wedge \mathrm{d}\boldsymbol{r}^\beta \cdot \frac{\boldsymbol{r}^\alpha - \boldsymbol{r}^\beta}{|\boldsymbol{r}^\alpha - \boldsymbol{r}^\beta|^3} \tag{2}$$

is the Gaussian linking number. As discussed in the literature [1,2], the double line integral on the rhs is a topological invariant, i.e. is invariant with respect to all deformations of the rings given by their coordinates $\{\boldsymbol{r}^\alpha\}$ and $\{\boldsymbol{r}^\beta\}$ which do not alter the topological state. However, this invariant is not unique. A counterexample is the Whitehead link [1] which represents a linked state between two rings and has the linking number $\Phi(C_\alpha, C_\beta) = 0$, the same value which results for a pair of unlinked rings. However, the Whitehead link is maintained by the self-interaction of one ring with itself. For random walk rings, these self-interactions effectively disappear, and thus the Whitehead link is unknotted to a pair of unlinked rings.

The partition function of the system with a given linking number n reads as

$$Z(n) = Z_0 \left\langle \delta\left(n, \sum_{\beta\neq 1}^{n_p} \Phi(C_1, C_\beta)\right)\right\rangle_{\{\boldsymbol{r}^1(s)\},\{\boldsymbol{r}^\beta(s)\}}. \tag{3}$$

The Kronecker delta appearing in equation (3) may be expressed in terms of the integral

$$\delta(n, f) = \int_{-\pi}^{\pi} \frac{\mathrm{d}g}{2\pi} \mathrm{e}^{\mathrm{i}gn - \mathrm{i}gf} \tag{4}$$

giving rise to a topological 'charge' g conjugate to n. One obtains the corresponding conjugate partition function $Z(g)$

$$Z(g) = \left\langle \exp\left(\mathrm{i}g \sum_{\beta\neq 1}^{n_p} \Phi(C_1, C_\beta)\right)\right\rangle_{\{\boldsymbol{r}^1(s)\},\{\boldsymbol{r}^\beta(s)\}}. \tag{5}$$

The parameter g may therefore be considered to be the chemical potential for the linking numbers. In what follows, the conformational model for the rings is restricted to closed random walk chains. The excluded volume effect is neglected. As discussed in [17], this simplification restores the uniqueness of the Gauss invariant for the links.

The exponential function in equation (5) may now be expressed (in the spirit of a Hubbard–Stratonovich transformation) in terms of an Abelian BF theory as follows [22]:

$$Z(g) = \mathcal{N} \int \mathcal{D}\boldsymbol{A} \int \mathcal{D}\boldsymbol{B} \exp\left(\mathrm{i} \int \boldsymbol{B} \wedge \mathrm{d}\boldsymbol{A}\right) \times \left\langle \exp\left(\mathrm{i}g \oint_{C_1} \mathrm{d}\boldsymbol{r}^1 \cdot \boldsymbol{A}\right)\right\rangle_{\{\boldsymbol{r}^1(s)\}} \left\langle \exp\left(\mathrm{i} \int \boldsymbol{B} \cdot \boldsymbol{j}\right)\right\rangle_{\{\boldsymbol{r}^\beta(s)\}}. \tag{6}$$

The constant $\mathcal{N}$ is a normalization. The wedge product $\wedge$ is an exterior product which generates the algebra of differential forms over $\mathbb{R}^3$, $\Omega^*(\mathbb{R}^3)$ [24]. In this language $\boldsymbol{A}$ and $\boldsymbol{B}$ are one-forms,

and the invariant expression $\boldsymbol{B} \wedge \mathrm{d}\boldsymbol{A}$ is a three-form. In terms of local coordinates it reads as $\epsilon_{\lambda\mu\nu} B_\lambda \partial_\mu A_\nu \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. Therefore, no volume element appears in the integration. Line integrations are specified explicitly. The notation given above has been widely used in field theory literature [22], and is used here to make manifest the coordinate invariance of the field theory. Later on, when evaluating two-point functions, we return to explicit coordinate representations of the fields. The variable $\boldsymbol{j}(\boldsymbol{x})$ is the tangent vector density, simply called the 'current', of the chains $\beta = 2 \ldots n_p$

$$\boldsymbol{j}(\boldsymbol{x}) = \sum_\beta \oint_0^N \mathrm{d}s \, \dot{\boldsymbol{r}}^\beta(s) \delta(\boldsymbol{x} - \boldsymbol{r}^\beta(s)). \tag{7}$$

The conformational averages with respect to the test ring C_1 and with respect to the other rings factorizes for random walk chains. The conformation of the surrounding rings are represented in terms of the current $\boldsymbol{j}$. In fact, they have fused to a single effective ring. As a reminder, the appearance of self-linking numbers has been avoided.

Next, the conformational coordinates for the surrounding rings C_1 for $\beta \neq 1$ are summed. Then, the functional integral with respect to the gauge field $\boldsymbol{B}$ is performed. First, the average with respect to the ring chains C_β is found. For random walk rings one obtains

$$\left\langle \exp\left(\mathrm{i} \int \boldsymbol{B} \cdot \boldsymbol{j} \right) \right\rangle_{\{\boldsymbol{r}^\beta(s)\}} = \exp\left(-\tfrac{1}{2} \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{x}' \, \boldsymbol{B}_\mu(\boldsymbol{x}) \boldsymbol{B}_\nu(\boldsymbol{x}') \langle \boldsymbol{j}_\mu(\boldsymbol{x}) \boldsymbol{j}_\nu(\boldsymbol{x}') \rangle \right). \tag{8}$$

On the rhs of this equation the chain index has been omitted. The average of the correlation function for the currents at points $\boldsymbol{x}$ and $\boldsymbol{x}'$ is carried out in the limit of very long rings. Let us denote the number of segments per ring by N and the average segment length by l. Neglecting $1/N$ terms arising due to the closure constraint [28] for the surrounding rings, the current–current correlation function is given by the following expression (the same approximation is made in [23])

$$\langle \boldsymbol{j}_\mu(\boldsymbol{x}) \boldsymbol{j}_\nu(\boldsymbol{x}') \rangle = \delta_{\mu\nu} \frac{\rho l^2}{d} \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{9}$$

where $\rho = (n_p N)/V$ is the average segment density. Equation (9) is an approximation valid for sufficiently concentrated systems. The question may be raised for these systems whether the Gauss invariant is still valid. Certainly the probability for more complicated links, such as the Borromean rings (a three-ring link which falls apart when cutting one ring), which are only detected by HOLC, is higher but remains low compared to the Gauss invariant measuring the pairwise entanglement of rings. In order to determine the second topological moment given in the next section, these considerations are irrelevant as the Gauss invariant is simply an observable of the system (see below).

Let $G = \rho l^2/3$ and then the partition function reads as

$$Z(g) = \mathcal{N} \int \mathcal{D}\boldsymbol{A} \int \mathcal{D}\boldsymbol{B} \exp\left(\mathrm{i} \int \boldsymbol{B} \wedge \mathrm{d}\boldsymbol{A} - \frac{G}{2} \int \boldsymbol{B} \cdot \boldsymbol{B} \right) \left\langle \exp\left(\mathrm{i}g \oint_{C_1} \mathrm{d}\boldsymbol{r}^1 \cdot \boldsymbol{A} \right) \right\rangle_{\{\boldsymbol{r}^1(s)\}} \tag{10}$$

where $\mathcal{N}$ is an adjusted normalization factor. Now integration with respect to the gauge fields $\boldsymbol{B}$ may be carried out, and this yields the effective model of Brereton and Shah [23]

$$Z(g) = \mathcal{N} \int \mathcal{D}\boldsymbol{A} \exp\left(-\frac{1}{2G} \int (\nabla \wedge \boldsymbol{A})^2 \right) \left\langle \exp\left(\mathrm{i}g \oint_{C_1} \mathrm{d}\boldsymbol{r}^1 \cdot \boldsymbol{A} \right) \right\rangle_{\{\boldsymbol{r}^1(s)\}}. \tag{11}$$

Leaving aside for the moment the average with respect to the C_1 ring and using the language of quantum field theory, one is left with a so-called Wilson loop that is averaged with respect to

a Euclidean Yang–Mills theory in 2 + 1 dimensions. Wilson loops are considered in quantum field theory for studying the confinement problem. An analogy to this problem has been used in [25] to study the collapse transition of randomly entangled polymer rings in two-dimensions (for earlier studies see [26, 27]).

3. The second topological moment

The second topological moment $\langle n^2 \rangle$ for a single test ring entangled with many other rings given the restriction that all conformational averages are taken with respect to random walks may now be determined rather simply. Following [20, 21] $\langle n^2 \rangle$ may be determined from

$$\langle n^2 \rangle = -\frac{\partial^2}{\partial g^2} Z(g)|_{g=0}. \tag{12}$$

More generally $Z(g)$ is the generating function for all topological moments. It is formulated for the present purpose as follows:

$$Z(g) = \left\langle \exp\left(\mathrm{i}g \oint_{C_\alpha} \mathrm{d}\boldsymbol{r}^\alpha \boldsymbol{A} \right) \right\rangle_{\{\boldsymbol{A}\},\{\boldsymbol{r}^\alpha(s)\}}. \tag{13}$$

The normalization factor has been absorbed into the average with respect to $\boldsymbol{A}$. To separate the gauge fields and the conformational coordinates, the line integral in equation (13) is expressed as follows:

$$\oint_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{A} = \int_{\boldsymbol{k}} A_\mu(\boldsymbol{k}) \oint_C \mathrm{d}s\, \dot{r}_\mu(s) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}(s)}. \tag{14}$$

The summation convention is understood. The abbreviation $\int_{\boldsymbol{k}}$ represents $\int \mathrm{d}^3k/(2\pi)^3$. In order to perform the functional integral with respect to $\boldsymbol{A}$, a gauge has to be used, which in the present case is $\nabla \cdot \boldsymbol{A} = 0$. In this case the two-point correlation function for $\boldsymbol{A}$ in Fourier space is given by

$$\langle A_\mu(\boldsymbol{k}) A_\nu(\boldsymbol{q}) \rangle_{\{\boldsymbol{A}\}} = \delta(\boldsymbol{k} + \boldsymbol{q}) \frac{G}{k^2} (\delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\nu) \tag{15}$$

where $\hat{k}_\mu = k_\mu/|\boldsymbol{k}|$ are the components of a unit vector. In order to perform the conformational average, the only expression needed is

$$\Phi_{\mu\nu}(s_1, s_2; \boldsymbol{k}) = \langle \dot{r}_\mu(s_1) \dot{r}_\mu(s_2) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}(s_1) - \boldsymbol{r}(s_2))} \rangle. \tag{16}$$

This average has been evaluated for random walk rings [28] and reads as follows:

$$\begin{aligned} \Phi_{\mu\nu}(s_1, s_2; \boldsymbol{k}) = &\left[\frac{l^2}{3} \delta_{\mu\nu} \left(\delta(s_1 - s_2) - \frac{1}{N} \right) + \frac{l^4}{9N} k_\mu k_\nu |s_1 - s_2| \left(1 - \frac{|s_1 - s_2|}{N} \right) \right] \\ &\times \exp\left[-\frac{l^2 k^2}{6} |s_1 - s_2| \left(1 - \frac{|s_1 - s_2|}{N} \right) \right]. \end{aligned} \tag{17}$$

The k-dependent terms in the first factor on the rhs of this equation do not give any contributions due to gauge invariance. This is most obvious in the Landau gauge, $\boldsymbol{k} \cdot \boldsymbol{A}(\boldsymbol{k}) = 0$.

Now, the second topological moment is given by (using a summation convention for repeated indices)

$$\langle n^2 \rangle = -\left\langle \int_{\boldsymbol{k}} \int_{\boldsymbol{q}} A_\mu(\boldsymbol{k}) A_\nu(\boldsymbol{q}) \oint \mathrm{d}s_1 \oint \mathrm{d}s_2\, \dot{r}_\mu(s_1) \dot{r}_\nu(s_2) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}(s_1)} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}(s_2)} \right\rangle_{\{\boldsymbol{A}\},\{\boldsymbol{r}(s)\}}. \tag{18}$$

Using the correlators (15) and (17), one obtains the expression

$$\langle n^2\rangle = G\int_{\boldsymbol{k}}\frac{1}{k^2}\oint \mathrm{d}s_1\oint \mathrm{d}s_2\left[\frac{l^2}{3}(\delta_{\mu\mu}-1)\left(\delta(s_1-s_2)-\frac{1}{N}\right)\right] \times\exp\left[-\frac{l^2k^2}{6}|s_1-s_2|\left(1-\frac{|s_1-s_2|}{N}\right)\right]. \tag{19}$$

The last equation gives rise to two terms. The first stems from the delta function $\delta(s_1-s_2)$ and is readily evaluated. The second needs further analysis: if $s_1=s_2$, the exponential equals 1, and one needs to evaluate an integral in $\boldsymbol{k}$ space which depends on a cutoff $\Lambda\sim\xi^{-1}$; if $s_1\neq s_2$, the integration with respect to s_1, s_2 gives a non-trivial contribution to the integration in $\boldsymbol{k}$-space, the details of which are discussed in the appendix. When inserting for G the result is

$$\langle n^2\rangle = \rho l^3\left(\frac{Nl\Lambda}{9\pi^2}-\frac{\sqrt{N}}{\sqrt{2(3\pi)^3}}-\frac{l\Lambda}{9\pi^2}\right) \tag{20}$$

where $\Lambda=\xi^{-1}$ is an inverse cutoff length. As the result has been obtained in the limit $N\gg 1$, the last term which is independent of N may be dropped, so the final result in the large N limit reads as

$$\langle n^2\rangle \simeq \rho l^3\left(\frac{Nl\xi^{-1}}{6\pi^2}-\frac{\sqrt{N}}{\sqrt{3(2\pi)^3}}\right). \tag{21}$$

Apart from the numerical prefactors, the result is identical to the one derived from an n-component field theory for the limit $n\to 0$ in [20]. The leading scaling term in equation (21) was also found by Brereton and Shah [29] who calculated the generating function $Z(g)$ directly in the conformational space without the use of gauge fields, using a pre-averaging procedure as a further approximation. Their result contains, however, the square root of the ratio l/ξ in contrast to the present result which is a minor difference and might be due to the approximation used in their work.

4. Discussion

The topological constraint of the fixed linking number of a test ring entangled with n_p surrounding rings has been implemented within the framework of a random walk model for polymer conformation using the simplest link invariant, the Gauss integral. This was reformulated in terms of a topological quantum field theory: the so-called Abelian BF theory. Two results were obtained: first, the model of Brereton and Shah [23] was derived, second, the second topological moment for the limit of large segment numbers $N\gg 1$ was calculated, in agreement with the previous result by Ferrari *et al* [20, 21]. The same result may also be obtained from directly averaging the squared Gauss integral with respect to a random walk conformation, a work which will be presented elsewhere. The gauge field approach, however, is conceptually very appealing as it separates the topological interactions from conformational entropy.

The present method avoids the complicated n-component field theory used in [20]. The latter approach is, however, better suited for treating the complete problem including excluded volume interactions. On the other hand, the Gauss invariant is ambiguous for self-avoiding walk rings. After considering work by Moroz and Kamien [31] the problem remains whether a topological field coupled to the n-component field for the polymer conformation changes the random walk result significantly. The authors have considered self-avoiding walks with writhe. In fact, a chemical potential for writhe is introduced which gives the coupling constant

for the interaction between a topological gauge field and a conformational n-component field, in a very similar way to that in [20]. The scaling behaviour of the radius of gyration and the first two moments of writhe are calculated. As to the first question, the effect of writhe is found to be irrelevant (to one-loop), i.e. the self-avoiding walk fix point corresponding to the exponent $\nu \simeq 0.588$ remains unchanged. Essentially this result is due to the fact that a topological field theory lacks a scale. With respect to the moments of the writhe (which is similar to calculating moments of n as done above), no scaling dependence on the number of polymer segments is found which is a surprising result. A clarification of this issue is certainly necessary. Let us note that Kholodenko and Vilgis [32] calculated the writhe of semiflexible polymers which was found to scale as $N^{1/2}$.

Finally let us interpret the results of equation (21) in simple terms. Obvious the second topological moment basically scales as $\langle n^2 \rangle \sim \rho l^3 N$ as N becomes large. A simple argument, also discussed in [30], gives a similar result. Let the average segment density ρ inside the test ring be given by the number of rings crossing the interior of the test ring, which is equal to the average linking number per ring $\bar{n}$, times the number of segments N, divided by the volume of the test ring R^3. Then the density reads as

$$\rho = \frac{\bar{n}N}{R^3}. \tag{22}$$

Assuming that the segment density in the interior of the test ring is equal to the segment density elsewhere in the system, and employing $R^3 \sim l^3 N^{3/2}$, i.e. using the Gaussian result for the ring volume, one immediately obtains

$$\bar{n} = \rho l^3 \sqrt{N}. \tag{23}$$

Both $\bar{n}$ and $\sqrt{\langle n^2 \rangle}$ scale as $\sqrt{N}$, so the theoretical calculation presented above reproduces the characteristic scaling with the chain length derived from this simple argument. The difference between the density-dependent prefactors appears to be superficial. In fact in the case of dense melts where $\rho \sim l^{-3}$ and which is the regime where the above argument applies, the prefactors coincide and $\bar{n} = \sqrt{\langle n^2 \rangle}$.

Acknowledgments

Discussions with F Ferrari and H Kleinert are gratefully acknowledged. The author is grateful to T A Vilgis who introduced him to the subject of polymer topology. The author thanks M Weigt for a critical reading of the paper, and acknowledges financial support by the DFG under grant no Zi209/6-1.

Appendix

Here the details of the calculation of the rhs of equation (19)

$$\langle n^2 \rangle = \frac{2}{3} G \int_k \frac{1}{k^2} \oint \mathrm{d}s_1 \oint \mathrm{d}s_2 \left[l^2 \left(\delta(s_1 - s_2) - \frac{1}{N} \right) \right] \times \exp\left[-\frac{l^2 k^2}{6} |s_1 - s_2| \left(1 - \frac{|s_1 - s_2|}{N} \right) \right] \tag{24}$$

is given in more detail. The delta function in the integrand gives rise to the integral

$$I_1 = \frac{2}{3} G N l^2 \int_k \frac{1}{k^2} = \frac{2}{3} G N l^2 \frac{4\pi}{(2\pi)^3} \int_0^\Lambda \mathrm{d}k = G N l^2 \frac{\Lambda}{3\pi^2} \tag{25}$$

where the $\boldsymbol{k}$ integration has been carried out up to the cutoff parameter $\Lambda \sim \xi^{-1}$. The $1/N$ term in the integrand on the rhs of equation (24) is given by

$$I_2 = -\frac{2}{3} G \frac{l^2}{N} \int_{\boldsymbol{k}} \frac{1}{k^2} \oint \mathrm{d}s_1 \oint \mathrm{d}s_2 \exp\left[-\frac{l^2 k^2}{6} |s_1 - s_2| \left(1 - \frac{|s_1 - s_2|}{N} \right) \right]. \tag{26}$$

Concerning the integration with respect to s_1 and s_2 in the last equation, two cases need to be considered: $s_1 = s_2$ and $s_1 \neq s_2$. In the first case, the exponential function gives 1 and the integration gives a factor of N, the result being

$$I_{2a} = -G l^2 \frac{\Lambda}{3\pi^2}. \tag{27}$$

The case $s_1 \neq s_2$ is slightly more involved. First, let us integrate with respect to k, which gives

$$I_{2b} = -G \frac{l^2}{N} \frac{4\pi}{(2\pi)^3} \oint \mathrm{d}s_1 \oint \mathrm{d}s_2 \sqrt{2\pi} \left[\frac{l^2}{3} |s_1 - s_2| \left(1 - \frac{|s_1 - s_2|}{N} \right) \right]^{-1/2}. \tag{28}$$

Now the domain of integration with respect to s_1 and s_2 may be restricted to two times one half that of the original one giving

$$I_{2b} = -\frac{2}{3} G \frac{l^2}{N} \frac{4\pi}{(2\pi)^3} 2 \int_0^N \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \sqrt{2\pi} \left[\frac{l^2}{3} (s_1 - s_2) \left(1 - \frac{(s_1 - s_2)}{N} \right) \right]^{-1/2}. \tag{29}$$

A shift of variables $u = (s_1 - s_2)/N$, $v = (s_1 + s_2)/N$ is easily performed and leads to

$$I_{2b} = -\frac{2}{3} G \frac{l^2}{N} \frac{4\pi}{(2\pi)^3} \sqrt{2\pi} N^2 \int_0^1 \mathrm{d}u \int_a^{2-a} \mathrm{d}v \left[\frac{l^2}{3} N u (1 - u) \right]^{-1/2}. \tag{30}$$

From now on the integration is elementary and one obtains

$$I_{2b} = -G l N^{1/2} \frac{1}{\sqrt{6\pi^3}}. \tag{31}$$

Adding I_1, I_{2a}, and I_{2b} and inserting $G = \rho l^3/d$ for $d = 3$ gives the rhs of equation (19) in the main text.

References

[1] Kauffman L H 1993 *Knots and Physics* (Singapore: World Scientific)
[2] Guadagnini E 1993 *The Link Invariants of the Chern–Simons Field Theory* (Berlin: de Gruyter)
[3] Gilbert N D and Porter T 1994 *Knots and Surfaces* (Oxford: Oxford University Press)
[4] Kleinert H 1995 *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics* 2nd edn (Singapore: World Scientific)
[5] Nechaev S K 1996 *Statistics of Knots and Entangled Random Walks* (Singapore: World Scientific)
[6] Cotta-Ramusino P, Guadagnini E, Martellini M and Mintchev M 1990 *Nucl. Phys.* B **330** 557
Guadagnini E, Martellini M and Mintchev M 1990 *Nucl. Phys.* B **330** 575
[7] Kholodenko A L and Vilgis T A 1998 *Phys. Rep.* **298** 251
[8] Wasserman E 1960 *J. Am. Chem. Soc.* **82** 4433
[9] Frisch H L and Wasserman E 1962 *J. Am. Chem. Soc.* **83** 3789
[10] Edwards S F 1968 *J. Phys. A: Math. Gen.* **1** 15
[11] Chern S S and Simons J 1974 *Ann. Math.* **99** 48
[12] Witten E 1989 *Commun. Math. Phys.* **121** 351
[13] Kholodenko A L and Vilgis T A 1994 *J. Physique (Paris)* **4** 843
[14] Axelrod S, Della Pietra S and Witten E 1991 *J. Diff. Geom.* **33** 787
[15] Alvarez M and Labastida J M F 1993 *Nucl. Phys.* **395** 198
Alvarez M and Labastida J M F 1995 *Nucl. Phys.* **433** 555
[16] Brereton M G and Vilgis T A 1995 *J. Phys. A: Math. Gen.* **28** 1149
[17] Otto M 1996 Die statistische Mechanik flexibler, verschlaufter und unverschlaufter Polymerringe *PhD Thesis* University of Mainz, available at http://www.Theorie.Physik.UNI-Goettingen.DE/~otto/pubs.html

[18] Otto M and Vilgis T A 1998 *Phys. Rev. Lett.* **80** 881
[19] Ferrari F and Lazzizzera I 1999 *Nucl. Phys.* B **539** 673
Ferrari F and Lazzizzera I 1998 *Phys. Lett.* B **444** 167
[20] Ferrari F, Kleinert H and Lazzizzera I 2000 *Phys. Lett.* A **276** 31
(Ferrari F, Kleinert H and Lazzizzera I 2000 *Preprint* cond-mat/0002049)
Ferrari F, Kleinert H and Lazzizzera I 2000 *Eur. Phys. J.* B **18** 645
(Ferrari F, Kleinert H and Lazzizzera I 2000 *Preprint* cond-mat/0003355)
[21] Ferrari F, Kleinert H and Lazzizzera I 2000 Field theory of N entangled polymers *Preprint* cond-mat/0005300
[22] Blau M and Thompson G 1991 *Ann. Phys., NY* **205** 130
[23] Brereton M G and Shah S 1980 *J. Phys. A: Math. Gen.* **13** 2751
[24] Nakahara M 1990 *Geometry, Topology and Physics* (Bristol: Adam Hilger)
[25] Otto M and Vilgis T A 1996 *J. Phys. A: Math. Gen.* **29** 3893
[26] Nechaev S K and Rostiashvili V G 1993 *J. Physique II (France)* **3** 91
[27] Rostiashvili V G, Nechaev S K and Vilgis T A 1993 *Phys. Rev.* E **48** 3314
[28] Brereton M G and Vilgis T A 1992 *Phys. Rev.* A **45** 7413
[29] Brereton M G and Shah S 1982 *J. Phys. A: Math. Gen.* **15** 985
[30] Vilgis T A and Otto M 1997 *Phys. Rev.* E **56** R1314
[31] Moroz J D and Kamien K D 1997 *Nucl. Phys.* B **506** 695
[32] Kholodenko A L and Vilgis T A 1996 *J. Phys. A: Math. Gen.* **29** 939